Problem 2.

(1) Decide, whether a set $[x, y, z] \in \mathbb{R}^3$ which satisfies

$$e^{xy} + xyz = e + 1 \tag{1}$$

can be described on some neighborhood of point $[1, 1, 1]$ as a graph of a function $x = x(y, z) : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\infty$, which is defined on some neighborhood of $[1, 1]$ satisfying $x(1, 1) = 1$.

(2) Similarly as in (1) decide, whether the set can be described as a graph of $\mathcal{C}^\infty$ function $y = y(x, z)$, which is defined on some neighborhood of $[1, 1]$ satisfying $y(1, 1) = 1$.

(3) Compute $\frac{\partial^2 x}{\partial z \partial y}$ and $\frac{\partial^2 x}{\partial y \partial z}$.

(4) Determine a tangent plane to the graph of function y at point $[1, 1, 1]$.

If you use the implicit function theorem, then verify its conditions.

Solution

(1) We are going to verify conditions of the implicit function theorem for equation

$$F(x, y, z) = e^{xy} + xyz - e - 1 = 0$$

and point $[1, 1, 1]$.

- Clearly, $F \in \mathcal{C}^\infty(\mathbb{R}^3)$,
- $F(1, 1, 1) = 0$,
- $\frac{\partial F}{\partial x}(1, 1, 1) = e + 1 \neq 0$.

Thus $x(y, z)$ exists and belongs to $\mathcal{C}^\infty$.

(2) Similarly, like in (1). The only additional condition that needs to be verified is: $\frac{\partial F}{\partial y}(1, 1, 1) = e + 1 \neq 0$. Thus $y(x, z)$ exists and belongs to $\mathcal{C}^\infty$.

(3) Since $x \in \mathcal{C}^2$, we have $\frac{\partial^2 x}{\partial z \partial y} = \frac{\partial^2 x}{\partial y \partial z}$. Similarly, the derivative does not change, if we switch order of partial differentiation, when we compute second partial derivatives of function F. Using chain rule we obtain

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial y \partial z} F(x(y, z), y, z) = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\right) \\ &= \frac{\partial x}{\partial y} \cdot \left(\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 F}{\partial x \partial z}\right) + \frac{\partial F}{\partial x} \cdot \frac{\partial^2 x}{\partial y \partial z} + \frac{\partial^2 F}{\partial y \partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 F}{\partial y \partial z}. \end{aligned} \tag{2}$$

We compute

$$
\begin{aligned}
\frac{\partial F}{\partial x}(x,y,z) &= ye^{xy} + yz, & \frac{\partial F}{\partial x}(1,1,1) &= e+1,\\
\frac{\partial F}{\partial y}(x,y,z) &= xe^{xy} + xz, & \frac{\partial F}{\partial y}(1,1,1) &= e+1,\\
\frac{\partial F}{\partial z}(x,y,z) &= xy, & \frac{\partial F}{\partial z}(1,1,1) &= 1,\\
\frac{\partial^2 F}{\partial x^2}(x,y,z) &= y^2e^{xy}, & \frac{\partial^2 F}{\partial x^2}(1,1,1) &= e,\\
\frac{\partial^2 F}{\partial x\partial y}(x,y,z) &= (xy+1)e^{xy} + z, & \frac{\partial^2 F}{\partial x\partial y}(1,1,1) &= 2e+1,\\
\frac{\partial^2 F}{\partial x\partial z}(x,y,z) &= y, & \frac{\partial^2 F}{\partial x\partial z}(1,1,1) &= 1,\\
\frac{\partial^2 F}{\partial y\partial z}(x,y,z) &= x, & \frac{\partial^2 F}{\partial y\partial z}(1,1,1) &= 1.
\end{aligned}
$$

By (2) we have

$$
\begin{aligned}
\frac{\partial x}{\partial y}(1,1) &= -\frac{\frac{\partial F}{\partial y}(1,1,1)}{\frac{\partial F}{\partial x}(1,1,1)} = -1,\\
\frac{\partial x}{\partial z}(1,1) &= -\frac{\frac{\partial F}{\partial z}(1,1,1)}{\frac{\partial F}{\partial x}(1,1,1)} = -\frac{1}{e+1},\\
0 = -1\left(e\cdot\frac{-1}{e+1}+1\right) &+ (e+1)\frac{\partial^2 x}{\partial y\partial z}(1,1) + (2e+1)\cdot\frac{-1}{e+1} + 1.
\end{aligned}
$$

Thus $\frac{\partial^2 x}{\partial y\partial z}(1,1) = \frac{1}{e+1}$.

(4) Formula for a tangent plane to the graph of function y at point $[1,1,1]$ ($T(x,z)=y$) is same like formula for a tangent plane to the graph of function x at point $[1,1,1]$:

$$
\frac{\partial F}{\partial x}(1,1,1)(x-1) + \frac{\partial F}{\partial y}(1,1,1)(y-1) + \frac{\partial F}{\partial z}(1,1,1)(z-1) = 0.
$$

Thus

$$
T(x,z) = -x + 2 - \frac{z-1}{e+1}.
$$